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Lambert series and Liouville's identities

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Abstract

The relationship between Liouville's arithmetic identities and products of Lambert series is investigated. For example it is shown that Liouville's arithmetic formula for the sum

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=n}} (F(a-b) - F(a+b)),$$

where $n \in \mathbb{N}$ and $F : \mathbb{Z} \rightarrow \mathbb{C}$ is an even function, is equivalent to the Lambert series for

$$\left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin n\theta \right)^2 \quad (\theta \in \mathbb{R}, |q| < 1)$$

given by Ramanujan.

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